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CONSTRUCTION OF STRONGLY REGULAR GRAPHS, TWO-WEIGHT CODES AND PARTIAL GEOMETRIES BY FINITE FIELDS

by

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We give a general construction method for strongly regular graphs with $q=p^{(e-1)t}$ vertices and valency $k=u(q-1)/e$, where p and e ($\neq 2$) are prime numbers such that p is primitive (mod e), and u ($<e$) and t are natural numbers. These graphs are of so-called Latin square type (for odd t), and of negative Latin square type (for even t). Some of the graphs of the latter type are new. They give rise to new two-weight codes. Moreover we show the existence of a partial geometry with parameters $s=t=5$ and $\alpha=2$, i.e., a system of 81 points and 81 lines such that each point (line) is incident with exactly 6 lines (points), no two lines intersect in more than one point, and any point outside any line is collinear with exactly 2 points on that line. (This was the only unknown partial geometry with $s=t$ and $\alpha=2$.) The point graph of this geometry is a strongly regular graph with parameters $n=81$, $k=30$, and $\lambda=9$. Both constructions are based on choosing cyclotomic classes in finite fields.

1. Introduction

In this paper we describe a construction method for strongly regular graphs which produces some new graphs, and a related construction for a partial geometry which is also new. Both constructions are based on choosing cyclotomic classes in finite fields.

Although we remind the reader of a few definitions and theorems from the theory of strongly regular graphs (cf. [2]) we shall assume that this theory is known to him. We shall also make use of the theory of association schemes (cf. [2], [5]), and terminology and a few results from algebraic coding theory (cf. [9]).

Definition 1. A (simple, undirected) graph Γ is called *strongly regular*, with *parameters n, k, λ, μ*, if Γ has n vertices and

(i) Γ is regular with valency k;
(ii) if the vertices x and y are adjacent then there are exactly λ vertices adjacent to both x and y;
(iii) if the distinct vertices x and y are not adjacent then there are exactly μ vertices adjacent to both x and y.

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One easily checks that the complement of a strongly regular graph is strongly regular again. We shall exclude strongly regular graphs which are not connected since these are trivial. Similarly their complements are excluded. A graph Γ is described by its (0, 1) *adjacency matrix* A of size n defined by numbering the vertices and taking $a_{ij}=1$ iff the vertices i and j are adjacent. We quote the following theorems from the general theory of strongly regular graphs.

Theorem 1. *If Γ is a graph with n vertices and adjacency matrix A then Γ is strongly regular iff there are numbers k, r, s, μ such that $AJ=kJ$ and $(A-rI)(A-sI)=\mu J$.*

(J is the $n\times n$-matrix of ones.) We see that A has eigenvalues k (with multiplicity 1), r, and s. The multiplicities of r and s are denoted by f and g, respectively. The following theorem is a consequence of Theorem 1.

Theorem 2. *If Γ is a regular graph with adjacency matrix A and A has only three eigenvalues then Γ is a strongly regular graph.*

In this paper when discussing a strongly regular graph the symbols n, k, λ, μ, r, s, f, and g will denote the parameters described above ($r>s$). The strongly regular graphs constructed in this paper all have the elements of a finite field K as vertices, and for these graphs the additive group of K is a group of automorphisms of the graph. This allows us to use a theorem of Delsarte ([4]) concerning the relation between strongly regular graphs and two-weight codes.

Definition 2. (i) A linear code C over GF (q) is called a *projective code* if any two of its coordinates are linearly independent, i.e., if the dual code $C^{\perp}$ has minimum distance $\geqq 3$.

(ii) C is called a *two-weight code* if all non-zero code-words have weight w_1 or w_2 ($w_1<w_2$) for some w_1, w_2.

From a two-weight projective code C one obtains a strongly regular graph Γ by taking the code-words as vertices, and joining $\hat{x}$ and $\hat{y}$ by an edge iff the Hamming distance of $\hat{x}$ and $\hat{y}$ equals w_1. The verification of this fact is straightforward. We say Γ is *associated* with C.

Theorem 3. *If Γ is a strongly regular graph with parameters $n=p^d$, k, λ, μ, r, s, f, g for which the additive group of* GF (p^d) *acts as a regular group of automorphisms then Γ is the associated graph of a two-weight projective code over* GF (p) *of dimension d and word length N with weights w_1 and w_2 given by*

$$N=\frac{f}{p-1},\quad w_1=-\frac{(s+1)}{(r-s)}\cdot\frac{n}{p},\quad w_2=\frac{-s}{(r-s)}\cdot\frac{n}{p}.$$

In Section 2 we construct graphs using rank-3 groups. The constructions of Section 3 are generalizations obtained by slightly weakening some of the conditions imposed in Section 2.

2. Rank-3 Graphs

Let G be a permutation group acting on a set Ω. The group G has a natural action on the cartesian product $\Omega\times\Omega$. If G is transitive on Ω and has three orbits on $\Omega\times\Omega$ then G is called a *rank*-3 *group*. If G has even order then each of the two orbits O_1, O_2 different from the diagonal of $\Omega\times\Omega$ is symmetric. In that case the graph Γ with the elements of Ω as vertices and the pairs $\{p, q\}$ for which (p, q) and (q, p) are in O_1 as edges, is a strongly regular graph. Such graphs are called *rank*-3 *graphs.* To show that a given group of even order is a rank-3 group it is sufficient to show that G is transitive and that the stabilizer of a point has three orbits (on Ω).

We shall describe a rank-3 group which leads to a rank-3 graph Γ with the property that the complete graph on the same vertex set consists of a number of (edge-)disjoint graphs isomorphic to Γ and such that the union of any number of them is again a strongly regular graph. Some of the graphs which we construct appear to be new.

For the remainder of this section let p and e be primes, $e>2$ and p primitive (mod e). Let $t\in\mathbf{N}$ and $q=p^{(e-1)t}$. Consider the field $K=\mathrm{GF}(q)$ and let α be a primitive element of K. We denote the set $\{\alpha^{me}|0\leqq m<(q-1)/e\}$ by $K^{(e)}$. We define G to be the group of transformations $T_{a,b,\nu}$ of K given by

$$T_{a,b,\nu}(x):=ax^{p^\nu}+b \quad (a\in K^{(e)},\ b\in K,\ \nu\in\mathbf{Z}). \tag{2.1}$$

It is trivial to check that G is indeed a group.

Lemma 1. *G is a rank-3 group.*

Proof. (i) G is clearly transitive. (ii) $\{0\}$ and $\{\alpha^{me}|m\in\mathbf{N}\}$ are orbits of G_0. It remains to show that the remaining elements of K form an orbit of G_0. Let $x=\alpha^{me+i}$ $(0<i<e)$, $y=\alpha^{me+j}$ $(0<j<e)$. There is a ν such that $ip^\nu\equiv j$ (mod e) because p is primitive (mod e). This implies that $x^{p^\nu}/y\in K^{(e)}$ and hence there is an $a\in K^{(e)}$ such that $T_{a,0,\nu}(x)=y$. ∎

The strongly regular graph corresponding to G can be defined directly by taking the elements of K as vertices and joining x and y by an edge iff $x-y\in K^{(e)}$. Its valency is clearly $k:=(q-1)/e$.

Example 1. We treat a simple example extensively because it is amusing and also as an introduction to part of Section 3. Take $p=2$, $e=3$, $t=2$. Then $K=\mathrm{GF}(2^4)$. We find a strongly regular graph with parameters $n=16$, $k=5$, $\lambda=0$, $\mu=2$. It is known that this graph is unique (the so-called *Clebsch graph*). First we show that one of the usual representations of this graph (starting with the Petersen graph) can be obtained from the representation above. The sum of the elements of $K^{(3)}$ (i.e. $1+\alpha^3+\alpha^6+\alpha^9+\alpha^{12}$) is 0 and no proper subset of $K^{(3)}$ has zero sum. Therefore K consists of 0, the elements of $K^{(3)}$, and the ten sums of two distinct elements of $K^{(3)}$. Let $V:=\{0, 1, 2, 3, 4\}$ and identify a subset V' of V with $\{\alpha^{3i}|i\in V'\}$. Then Γ can be described by taking as vertices the subsets of V' with at most two elements, and joining two vertices by an edge iff their symmetric difference consists of one or four elements. (The subgraph on the ten two-element subsets is the Petersen graph.)

This representation leads to an idea which will be used again in Section 3. Let C be the binary code of length 5 consisting of all the words of even (Hamming-) weight. Take the elements of C as vertices of a graph and join two vertices by an edge iff their (Hamming-)distance is 4. The reader will easily verify that this is equivalent to the first representation of the Clebsch graph.

We now turn to the problem of determining the parameters λ and μ of Γ. What we use is essentially the theory of association schemes but the proof can be understood independently.

In the following A is the (0, 1) adjacency matrix of Γ. A character χ of (the additive group of) K will also be considered as a column vector (with positions indexed by the elements of K).

Lemma 2. *If χ is a character of K then χ is an eigenvector of A with eigenvalue* $r_\chi := \sum_{i=0}^{k-1} \chi(\alpha^{ei})$.

Proof. Since $A=(a_{xy})$, where $a_{xy}=1$ if $y-x\in K^{(e)}$ and $a_{xy}=0$ otherwise, we have

$$(A\chi)_x = \sum_{y\in K} a_{xy}\chi(y) = \sum_{i=0}^{k-1} \chi(x+\alpha^{ei}) = r_\chi \cdot \chi(x). \quad \blacksquare$$

Define the character χ_1 by

$$\chi_1(x) = e^{2\pi i\cdot \mathrm{Tr}(x)/p}. \tag{2.2}$$

It is a well-known fact that each character χ of K has a representation $\chi=\chi_a$, where $\chi_a(x):=\chi_1(ax)$, for $a\in K$. Since the vectors χ_a are mutually orthogonal Lemma 2 gives us all the eigenvectors of A. We now define a relation $\sim$ on the characters by

$$\chi' \sim \chi'' :\Leftrightarrow \exists a\in K^{(e)}\, \exists \nu\in \mathbf{Z}\, \forall x\in K[\chi'(x) = \chi''(ax^{p^\nu})]. \tag{2.3}$$

Clearly $\sim$ is an equivalence relation. In the same way as in the proof of Lemma 1 we see that the equivalence classes are $\{\chi_0\}$, $\{\chi_a | a\in K^{(e)}\}$, and $\{\chi_a | a\neq 0,\ a\notin K^{(e)}\}$. From (2.3) it follows that equivalent characters belong to the same eigenvalue. These eigenvalues are respectively k, $r_{\chi_1}=\sum_{i=1}^{k}\chi_1(\alpha^{ei})$, and $r_{\chi_\alpha}=\sum_{i=1}^{k}\chi_1(\alpha^{ei+1})$. Observe that $r_{\chi_\beta}=r_{\chi_\alpha}$ if $\beta=\alpha^j$, $1\leq j<e$. We have shown that A has (at most) three different eigenvalues, with multiplicities 1 (for the eigenvalue k), k, and $q-k-1$, respectively. It now immediately follows from the general theory of strongly regular graphs that the remaining parameters of Γ are given by

	λ	μ
t odd	$\dfrac{q-3e+1+(e-1)(e-2)\sqrt{q}}{e^2}$	$\dfrac{q-e+1-(e-2)\sqrt{q}}{e^2}$
t even	$\dfrac{q-3e+1-(e-1)(e-2)\sqrt{q}}{e^2}$	$\dfrac{q-e+1+(e-2)\sqrt{q}}{e^2}$

	r	s	f	g
t odd	$\dfrac{-1+(e-1)\sqrt{q}}{e}$	$\dfrac{-1-\sqrt{q}}{e}$	k	$q-k-1$
t even	$\dfrac{-1+\sqrt{q}}{e}$	$\dfrac{-1-(e-1)\sqrt{q}}{e}$	$q-k-1$	k

From the parameters we see that Γ is a Latin square graph if t is odd and that Γ is a negative Latin square graph, of type $NL_r(\sqrt{q})$ if t is even (cf. [8], [10]). Since the Latin square graphs are known the remaining results in this section are interesting for even t only. Therefore we restrict ourselves to the case of even t.

First we make the following trivial observation. For $0<j\leqq e$ we define the graph Γ_j on the elements of K as vertices, with an edge between two points x and y iff $x-y=\alpha^{ei+j}$ for some i (so $\Gamma=\Gamma_e$). Then all the graphs Γ_j are isomorphic and furthermore their edges partition the edges of the complete graph on q points. (Note that for the parameters of Example 1 this gives us a 3-colouring of the edges of K_{16} without monochromatic triangles, proving the result of Greenwood and Gleason [7] that the Ramsey number $R(3, 3, 3)$ is equal to 17 — cf. [13]). We aim to show that any union of a number of these graphs is again a strongly regular graph (however in general no longer a rank-3 graph).

Let A_j be the adjacency matrix of Γ_j. Then for any character χ of K we have

$$(A_j\chi)_x = \sum_{i=0}^{k-1} \chi(\alpha^{ei+j})\cdot\chi(x),$$

i.e. χ_a is an eigenvector of A_j, with eigenvalue k if $a=0$, r_{χ_1} if $a=\alpha^{ei-j}$ for some i, and r_{χ_α} otherwise. Let $J\subset\{1, 2, \ldots, e\}$. The matrix $A_J:=\sum_{j\in J} A_j$ is the adjacency matrix of the graph Γ_J on the elements of K as vertices, with an edge between x and y iff $y-x\in\alpha^j K^{(e)}$ for some $j\in J$. The eigenvalues of A_J are, if $u:=|J|$:

uk,	with multiplicity 1,
$r_{\chi_1}+(u-1)r_{\chi_\alpha}$,	with multiplicity uk,
ur_{χ_α},	with multiplicity $q-1-uk$.

Therefore the following theorem is a consequence of Theorem 2.

Theorem 4. *For any $J\subset\{1, 2, \ldots, e\}$ the graph Γ_J is a strongly regular graph with parameters*

$$n' = q, \quad k' = uk,$$

$$\lambda' = \frac{u^2q-3ue+u^2-(e-u)(e-2u)\sqrt{q}}{e^2},$$

$$\mu' = \frac{u^2q-ue+u^2+(eu-2u^2)\sqrt{q}}{e^2},$$

$$r' = \frac{-u+u\sqrt{q}}{e}, \quad s' = \frac{-u-(e-u)\sqrt{q}}{e},$$

$$f' = q-1-uk, \qquad g' = uk,$$

where $u:=|J|$.

(Note that t is even.) In Table I we list p, e, t (even), $u<\frac{1}{2}e$ and the parameters n, k, λ, μ ($n<1000$) of strongly regular graphs constructed in the manner described above.

Table I

p	e	t	u	n	k	λ	μ	Remarks
2	3	2	1	16	5	0	2	Clebsch graph
2	3	4	1	256	85	24	30	rank – 3
2	5	2	1	256	51	2	12	rank – 3
2	5	2	2	256	102	38	42	new
5	3	2	1	625	208	63	72	rank – 3, new

Remark 1. The matrices A_j are the adjacency matrices of an association scheme, namely a cyclotomic scheme (cf. [5]). Using the terminology of association schemes would not change the details of our proofs.

Remark 2. By Theorem 3 we now see that Theorem 4 yields new two-weight projective codes, and hence new 1-error correcting uniformly packed codes. By taking the complements of the graphs listed in Table I we find the following parameter sets of two-weight codes, three of which do not occur in the table of Van Tilborg [12].

Table II

alphabet size	word length	dimension	w_1	w_2	Remarks
2	5	4	2	4	code of Example 1
2	85	8	40	48	No. 3 with $r=5$ in [12]
2	51	8	24	32	
2	102	8	48	56	
5	52	4	40	45	

Remark 3. In this section we have required $e>2$. However, the results also hold for $e=2$, if $q\equiv 1 \pmod 4$, and in that case we find the well-known Paley graphs.

3. Graphs and Geometries Based on GF (3^d)

We remind the reader of the definition and some elementary properties of so-called partial geometries.

Definition 3. A *partial geometry*, with *parameters* s, t, α $(\geqq 1)$, is a finite incidence structure $(\mathscr{P}, \mathscr{B}, \mathrm{I})$ (the elements of $\mathscr{P}$ and $\mathscr{B}$ are called *points* and *lines*, respectively), where I is a symmetric incidence relation satisfying the following axioms:

(i) each point is incident with exactly $t+1$ lines;
(ii) each line is incident with exactly $s+1$ points;
(iii) two distinct points are incident with at most one line;
(iv) if x is a point and L is a line not incident with x, then there are exactly α points $x_1, \ldots, x_\alpha$ and α lines $L_1, \ldots, L_\alpha$ such that $x\mathrm{I}L_i$, $L_i\mathrm{I}x_i$ and $x_i\mathrm{I}L$ $(i=1, \ldots, \alpha)$.

We shall say that x and y are *collinear* ($x \sim y$) if x and y are incident with a line L. The following properties follow directly from Definition 3 by counting arguments (cf. [2], [11]).

For a partial geometry $(\mathscr{P}, \mathscr{B}, \mathrm{I})$ with parameters s, t, α we have:

(3.1)
(i) $v := |\mathscr{P}| = (s+1)(st+\alpha)/\alpha$;
(ii) $b := |\mathscr{B}| = (t+1)(st+\alpha)/\alpha$;
(iii) the graph with points as vertices and edges between collinear points is strongly regular, with parameters $n=v$, $k=s(t+1)$, $\lambda=s-1+t(\alpha-1)$, $\mu=(t+1)\alpha$.

Clearly (3.1) gives us a number of necessary conditions for the existence of a partial geometry. We are interested in the special case that $s=t$ and $\alpha=2$. It is easy to check that a partial geometry with these parameters can exist only if

$s=1$ (the geometry is a triangle),
$s=2$ (the geometry consists of the points and lines of AG (2, 3) from which one parallel class of lines has been removed), or
$s=5$.

We shall describe in two ways the construction of a partial geometry with $s=t=5$, $\alpha=2$ (this geometry appears to be new).

Construction 1. Let β be a primitive element of GF (3^4). Then $\gamma := \beta^{16}$ is a primitive 5-th root of unity. We consider the cyclic code C of length 5 over GF (3) defined by: $\hat{c} \in C \Leftrightarrow c(\gamma) = 0$. By the BCH-bound (cf. [9]) this code has minimum distance 5, i.e. it consists of $\hat{0}$, $\hat{1}$ and $\hat{2}$. This means that the elements of the set $S := \{0, 1, \gamma, \gamma^2, \gamma^3, \gamma^4\}$ have the property that if a linear combination of elements of S is 0 it must have the form

$$c_1 \cdot 0 + x_2 \cdot (1+\gamma+\gamma^2+\gamma^3+\gamma^4), \tag{3.2}$$

which, of course, follows trivially from the fact that γ is a primitive 5th root of unity. We consider $\mathscr{P}$:= the set of elements of GF (81), $\mathscr{B} := \{b+S \mid b \in \mathrm{GF}(81)\}$, and I is the inclusion relation, and claim:

Proposition. *$(\mathscr{P}, \mathscr{B}, \mathrm{I})$ is a partial geometry with $s=t=5$, $\alpha=2$.*

Proof. (i) Clearly every line has 6 points and every point is incident with 6 lines. Since, by (3.2), the differences of pairs of elements of S are all different a pair of points is incident with at most one line.

(ii) For the set of pairs (x, L) with $x \not\mathrm{I} L$ the average number of lines L_1 such that L_1 meets L and $L_1 \mathrm{I} x$, is 2. To show that this number is in fact 2 for every such pair (x, L) it is sufficient to consider (b, S), $b \not\mathrm{I} S$ and to show that there are at most 2 elements b_1, b_2 in S such that b and b_i are collinear. Now by definition b and b_1 are on a line iff $b = b_1 - b_1' + b_1''$ for some b_1' and b_2'' in S. This implies that b and b_1' are also collinear. If $b_1 = b_1'$ then $b = -b_1 - b_1''$ and then b and b_1'' are on a line, i.e. the equation $b = b_1 + b_1' - b_1''$ leads to exactly two points in S which are collinear with b. Clearly $b_1'' \neq b_1$. So if b is collinear with three or more elements of S we must have two relations $b = b_1 + b_1' - b_1''$ and $b = b_2 + b_2' - b_2''$. Now a little reflection shows that by subtracting them one cannot obtain the form (3.2). ∎

By (3.1) (iii) this partial geometry yields a strongly regular graph Γ with parameters $n=81$, $k=30$, $\lambda=9$, $\mu=12$. We have not been able to find a construction of a strongly regular graph with these parameters in the literature (although a construction of $NL_3(9)$ was announced in [10]).

In order to understand why this method works and to show a connection with the results of Section 2 we consider the following association scheme. We take the elements of GF (3^4) as points and call two elements x and y j-th associates iff $y-x=\beta^{8i+j}$ for some i $(1 \leq j \leq 8)$. In the notation of Section 2 we have taken $q=3^4$, $e=8$ (a divisor of $q-1$) and then x and y are j-th associates if $\{x, y\}$ is an edge of the graph Γ_j, which is no longer strongly regular. We find the eigenmatrix P of this association scheme by calculating $\sum_{i=0}^{9} \chi(\beta^{8i+j})$ (again using the notation of Section 2), for $j=1, 2, \ldots, 8$, where χ runs over the characters $\chi_1, \chi_\beta, \ldots, \chi_{\beta^7}$. Using GF (3^4) defined by $\beta^4=\beta+1$ we found:

$$P = \begin{pmatrix} 1 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\ 1 & 4 & 1 & 4 & -2 & -2 & 1 & -2 & -5 \\ 1 & 1 & 4 & -2 & -2 & 1 & -2 & -5 & 4 \\ 1 & 4 & -2 & -2 & 1 & -2 & -5 & 4 & 1 \\ 1 & -2 & -2 & 1 & -2 & -5 & 4 & 1 & 4 \\ 1 & -2 & 1 & -2 & -5 & 4 & 1 & 4 & -2 \\ 1 & 1 & -2 & -5 & 4 & 1 & 4 & -2 & -2 \\ 1 & -2 & -5 & 4 & 1 & 4 & -2 & -2 & 1 \\ 1 & -5 & 4 & 1 & 4 & -2 & -2 & 1 & -2 \end{pmatrix}.$$

The graph Γ following from Construction 1 can be described by: $\{x, y\}$ is an edge iff $y-x$ is the difference of two elements in S. Now S, and hence the set of differences is clearly closed under taking third powers, and furthermore all the elements β^{8i} occur among these differences. Then from $\beta^{16}-1=\beta^{73}$ it follows that Γ is apparently the union of Γ_1, Γ_3, and Γ_8. Indeed, if we add the columns of P corresponding to $j=1$, $j=3$, and $j=8$ we find only three eigenvalues, namely 30, 3, and -6 (which are the eigenvalues of Γ).

The discussion above shows that we can generalize the method of Section 2 as follows. Let β be a primitive element of GF (3^d). Let e be a divisor of 3^d-1. Form the eigenmatrix P of the association scheme on GF (3^d) (the *cyclotomic scheme*) where elements x and y are j-th associates if $y-x=\beta^{ei+j}$ for some i. If a subset of the columns of P has the property that in the sum of these columns only three different numbers occur, then by taking the union of the corresponding graphs Γ_j we find a strongly regular graph. In our example above it is obvious that Γ_4, Γ_5, and Γ_7 together yield a strongly regular graph Γ' isomorphic to Γ, the isomorphism being given by $\xi \to \beta^4\xi$, $\xi \in$ GF (3^4). By adding the columns corresponding to $j=2$ and $j=6$ we see that $\Gamma'':=\Gamma_2 \cup \Gamma_6$ is a strongly regular graph (of type $NL_2(9)$) with parameters $n=81$, $k=20$, $\lambda=1$, $\mu=6$. (For a geometric construction of such a graph we refer to C12 in [8].) From this discussion we see another generalization of the results of Section 2, namely the fact that the complete graph on 81 points is the edge-disjoint union of Γ, Γ', and Γ'', and that it is also the union of four graphs isomorphic to Γ''. Of course, the union of Γ_2, Γ_4, Γ_6, and Γ_8 is the Paley graph on 81 vertices. The union of Γ_1, Γ_2, Γ_5, and Γ_6 is also a strongly regular graph with the parameters of the Paley graph on 81 vertices. Both graphs are the point graphs of partial geometries with $s=8$, $t=4$, $\alpha=4$. However the two graphs are not isomorphic (we omit the tedious details). The graphs are rank-3 graphs (P. J. Cameron, oral communication). Our second construction uses a method similar to the method of Example 1 in Section 2.

Construction 2. Consider the dual code $C^\perp$ of the code in Construction 1. The code $C^\perp$ is defined by $\hat{c} \in C^\perp \Leftrightarrow (\hat{c}, \hat{1})=0$. In Table III we list the 81 code-words by weight and type, where e.g. type (1, 2, 0, 0, 0) means that the code-word has one coordinate 1 and one coordinate 2, and in the same way type (1, 1, 1, 0, 0) means that the non-zero coordinates are equal.

Table III

weight	type	number
0	(0, 0, 0, 0, 0)	1
2	(1, 2, 0, 0, 0)	20
3	(1, 1, 1, 0, 0)	20
4	(1, 1, 2, 2, 0)	30
5	(1, 1, 1, 1, 2)	10

We define three graphs $\bar{\Gamma}$, $\bar{\Gamma}'$, $\bar{\Gamma}''$ from $C^\perp$ by taking code-words as vertices and joining $\hat{x}$ and $\hat{y}$ by an edge in $\bar{\Gamma}$ if $d(\hat{x}, \hat{y})=2$ or 5, in $\bar{\Gamma}'$ if $d(\hat{x}, \hat{y})=4$, and in $\bar{\Gamma}''$ if $d(\hat{x}, \hat{y})=3$. ($d(\hat{x}, \hat{y})$ denotes the Hamming distance of $\hat{x}$ and $\hat{y}$.) Using Table III one easily checks that these graphs are strongly regular graphs, with the same parameters as Γ, Γ', and Γ'', respectively. In fact the graphs are isomorphic to these graphs. To provide the link with Construction 1 we consider the following description of $C^\perp$. Let γ be as in Construction 1. Then γ is a primitive 5-th root of unity in GF (3^4). Therefore

$$C^\perp = \{(\mathrm{Tr}\,(x), \mathrm{Tr}\,(x\gamma), \ldots, \mathrm{Tr}\,(x\gamma^4)) | x \in \mathrm{GF}\,(3^4)\}$$

(cf. [9] Theorem 3.4.3). Since Tr is a linear mapping we get a geometry isomorphic to the partial geometry of Construction 1 by taking the set of code-words corresponding to the set S as a line and again translating over all elements b in GF (81). This set is

$$\bar{S} = \{(0,0,0,0,0), (1,2,2,2,2), (2,1,2,2,2), (2,2,1,2,2), (2,2,2,1,2), (2,2,2,2,1)\}.$$

Clearly two code-words are adjacent in the graph $\bar{\Gamma}$, i.e. collinear in the geometry, iff their distance is 2 or 5.

From Theorem 3 and these constructions we have two two-weight projective codes over GF (3):

Table IV

word lenght	dimension	w_1	w_2
25	4	15	18
30	4	18	21

As was to be expected the second of these does not occur in the table of [12].

The reader familiar with Delsarte's work on association schemes (cf. [5] pp. 84—94) will have no trouble checking that the association schemes found from Construction 1 and Construction 2 are self-dual. In [5] Delsarte mentioned as an interesting example of a pair of dual strongly regular graphs the graphs with parameters $n=243$, $k=22$, $\lambda=1$, $\mu=2$ (constructed by Berlekamp, Van Lint and Seidel [1]), and with parameters $n=243$, $k=110$, $\lambda=37$, $\mu=60$ (found by Delsarte [5]), respectively. We remark that both of these graphs can be constructed using the method of this section. Take $q=3^5$, $e=11$ and form P as described above. The graph Γ_{11} turns out to be strongly regular (of course this follows from properties of the ternary Golay code). The other graph is a union of five of the graphs Γ_j. (This can be shown directly from properties of the Paley matrix of size 11.)

One is tempted to try to generalize the first construction to GF (3^6) in order to construct a partial geometry with $s=8$, $t=20$, $\alpha=2$. The lines of the geometry would be 9-cliques in the corresponding graph. Therefore we would need 21 sets $\beta^j K^{(91)}$ inside GF (3^6), where $K^{(91)}$ is the multiplicative group of the subfield GF (3^2). The requirement on the chosen exponents j to guarantee a partial geometry is that any four of the β^j span the field GF (3^6) as a vector space over GF (3^2). This means that we need a 3-arc in PG (2, 9). Since it has been shown by Cossu [3] that such an arc does not exist this approach cannot succeed.

Remark 4. Some of the examples in this section show that the idea of Section 2 sometimes works even if the requirements on e are relaxed. However, this is not true for the choice $p=7$, $e=3$, $t=1$, i.e. the assertion on p. 156 of [6] that one finds a strongly regular graph with parameters $n=49$, $k=16$, $\lambda=3$, on the elements of GF (49), with adjacency of x and y iff $x-y$ is a cube, is incorrect. In fact a strongly regular graph with these parameters does not exist.

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